Qualify Exam Math 708, 709
FALL 2018 ID number

Name: ________________________________

(Choose any 5 problems)

1. (20pts)

(a) Construct Lagrange's interpolation polynomial for the data given below.

(b) Construct Newton's interpolation polynomial for the data shown. Without simplifying it, write the polynomial obtained in nested form for easy evaluation.

x	1	-1	3	2	-2
y	0	1	-1	-5	3

2. (20pts) Let $f : R \to R$ be a C^2 function with a root x_* such that neither f' nor f'' has a root. Prove that Newton's method converges to x_* for any initial guess $x_0 \in R$, and show the convergence rate.

3. (20pts) Using Taylor series expansions, derive the following two formulas for approximating the third derivative. Find their error terms. Which formula is more accurate?

$$f'''(x) \approx \frac{1}{h^3}[f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)]$$

$$f'''(x) \approx \frac{1}{2h^3}[f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)]$$

4. (20pts) Let A be an invertible $n \times n$ matrix and let $\|\cdot\|$ denote the Euclidean norm.

(i) Define $cond(A)$, the conditional number of A.

(ii) Suppose that you solve do not solve $Ax = b$ exactly, but instead obtain an approximation, $\tilde{x}$, with nonzero residual $r = b - A\tilde{x}$. Derive an upper and lower bound for the error in $\tilde{x}$ relative to x in terms of $cond(A)$, $\|r\|$, and $\|b\|$.

(iii) Suppose now that A is orthogonal. What is $\|A\|$? Prove your claim. What is $cond(A)$? Prove your claim.

5. (20pts) Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$.

(i) Show that A can be written as $A = H + S$ where $H = H^\star$ is Hermitian and $S = -S^\star$ is skew-Hermitian. Give explicit formula for H and S in terms of A.

(ii) Show that $\sum_{i=1}^n |Re(\lambda_i)|^2 \leq \|H\|_F^2$, and $\sum_{i=1}^n |Im(\lambda_i)|^2 \leq \|S\|_F^2$.

(iii) Show that A is normal ($AA^\star = A^\star A$) if and only if $\sum_{i=1}^n |\lambda_i|^2 = \|A\|_F^2$.

6. (20pts)

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix}$$

(1) Find QR factorization of A

(2) To show that QR with shifts are more efficient than QR method for A.